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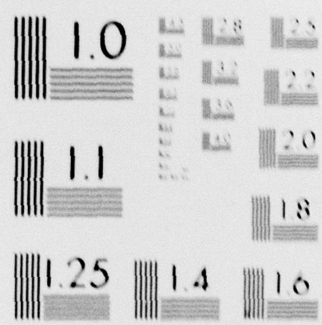
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REMARKS ON BIFURCATION THEORY IN DIFFERENTIAL EQUATIONS

by

Jack K. Hale

ABSTRACT

This paper discusses primarily the idea of genericity in bifurcation theory for one and two parameter problems. Recent results on homoclinic points are mentioned as well as a connection between the bifurcation function and dynamic behavior.

Remarks on Bifurcation Theory in Differential Equations

Jack K. Hale*

Abstract. This paper discusses primarily the idea of genericity in bifurcation theory for one and two parameter problems. Recent results on homoclinic points are mentioned as well as a connection between the bifurcation function and dynamic behavior.

My objective in this lecture is to point out some trends in the theory of bifurcation in differential equations and the manner in which very abstract theory has an influence on the basic understanding of specific examples. The examples will be in low dimension - never more than three. This choice is not made because examples never arise in higher dimensions. In fact, some of the most interesting applications of bifurcation theory today concern infinite dimensional systems - partial differential equations, functional differential equations or various types of integral equations. Even though the original problem is of higher dimension, the essential ingredient to bifurcation often is determined by a vector field in a low dimensional space. The appropriate space is obtained from the theory of integral manifolds

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which is discussed by Marsden [16] at this conference.

Even though I will not attempt to give an historical perspective on bifurcation theory, a careful study of the literature shows that Poincaré [26], [27] and Lyapunov [14], [15] are responsible for the basic philosophy as well as several of the fundamental ideas of the methods that we presently employ. One can find a direct link with the importance of exchanges of stability, the reduction principle to lower dimensional problems, the philosophy of genericity and the transformation theory so relevant to obtaining approximations of the center manifold and the flow on the center manifold. In many respects, we are still exploiting the ideas and methods of these two giants.

A fundamental step toward modern bifurcation theory in differential equations was made by Andronov and Pontryagin [3] when they gave a definition of structural stability in the plane. To avoid the difficulties that arise from the noncompactness of the plane, they restricted the discussion to the interior of a closed curve without contact to any of the vector fields to be considered. Two vector fields X,Y in C^k, $k \geq 1$, are <u>equivalent</u> if the trajectories of one are homeomorphic to the other. This is an equivalence relation among vector fields. X is <u>structurally stable</u> if every Y in a neighborhood of X is equivalent to X. The set of structurally stable systems is open and dense and characterized by the following properties: every critical point and periodic orbit is hyperbolic and no orbit connects saddles.

One can now say X is a <u>bifurcation point</u> (a vector field for which a perturbation could lead to a bifurcation) if X is structurally unstable; that is, not structurally stable. It is impossible to study the behavior of all vector fields in the neighborhood of an arbitrary bifurcation point; for example, $X = 0$ is a bifurcation point and any flow in the plane can be obtained by choosing an appropriate Y near zero. This is where the idea of genericity enters bifurcation theory. One must find those bifurcation points X which have the property that the simplest possible bifurcations occur near X. Andronov and Leontovich [1] made this precise by defining structural instability of degree k (or bifurcation point of degree k).

The vector field X is a bifurcation point of degree 0, if it is structurally stable. X is a bifurcation point of degree 1, if there is a neighborhood of X which has only bifurcation points of degree 0, or, ones which are equivalent to X. It is a bifurcation point of degree 2, if there is a neighborhood containing only bifurcation points of degree 0 or 1, or, ones which are equivalent to X. Similarly, one defines degree k.

It can be shown (see Andronov et al [2], Sotomayor [29]) that $k \geq 3$ implies X being a bifurcation point of degree 1 is equivalent to the fact that there is a neighborhood U of X such that $U = U_1 \cup \Gamma \cup U_2$ where Γ is a smooth submanifold of codimension one, U_1, U_2 are open sets belonging to distinct equivalence classes of structurally stable systems. Furthermore, X is a bifurcation point of degree 1, if and only if exactly one of the following alternatives hold as one passes through X from U_1 to U_2:

(i) Two limit cycles coalesce and disappear

(ii) A focus loses its hyperbolicity and a periodic orbit appears (elementary Hopf bifurcation)

(iii) A saddle and node coalesce and disappear

(iv) There is a smooth invariant curve containing a saddle and node which coalesce, disappear and the invariant curve becomes a periodic orbit.

(v) $\mathrm{tr}\,\partial X(0)/\partial x \neq 0$, a periodic orbit and saddle merge to form a homoclinic orbit (the stable and unstable manifolds of the saddle intersect) at X and then the homoclinic orbit disappears leaving only the saddle.

The fact that only two possibilities arise in a neighborhood of a bifurcation point of degree 1 suggests that this is the typical or generic situation that arises in the discussion of one parameter families of vector fields. Sotomayor [29] has proved this is the case. In fact, in the family of smooth one parameter families of vector fields in C^k, $k \geq 3$, the set which contains only bifurcation points of degree 0 or 1 is residual.

Extensive applications of these results to the theory of nonlinear oscillations was made in the late 1930's (see Andronov, Vitt and Khaikin [4]). Most of the results in the literature on one parameter problems are a consequence of these results.

The characterization of bifurcation points of degree two has recently been completed, but the detailed results will not be given here (see Andronov et al [2], Sotomayor [30], [31], Takens [32], Carr [5]). This theory of bifurcation points of degree two can be considered the typical or generic situation for two parameter families of vector fields. Genuine two parameter problems frequently arise in applications and often, the tendency is to scale the parameters in terms of a single small parameter in order to use some classical perturbation procedure. Sometimes it is possible (but with considerable effort) to show that the information obtained is exactly the same as one would obtain by allowing the original parameters to vary independently. In other cases, information is lost by scaling. The simplest illustration of this is the Hopf bifurcation when the degenerate term is fifth degree rather than cubic. For any one parameter family of vector fields for which the eigenvalues cross the imaginary axis with non-zero velocity, it can be shown that only one periodic orbit will be obtained (see Chafee [6]). If the eigenvalues are allowed to cross with zero velocity, two orbits sometimes appear (see Flockerzi [10]). If this problem is discussed in the general setting, no confusion or loss of information occurs. This result for Hopf bifurcation has recently been applied by Chow, et al [8].

Interesting applications of the two parameter methods for the equation

$$\ddot{x} + \lambda_1 x + \lambda_2 \dot{x} + f(x,\dot{x}) = 0$$

have been given by Takens [32], Holmes and Marsden [11], Carr [5] for the case when $f(x,0) = \beta x^3$, $\beta \neq 0$ and by Howard and Koppel [12] for $f(x,0) = \beta x^2$, $\beta \neq 0$.

For a discussion of bifurcation points of degree greater than two, see Andronov et al [2], Sotomayor [30], [31].

When the dimension of the system is >2, the class of structurally stable systems is small in the sense that it is not a residual set. In addition, many new phenomena occur which make a classification of such systems extremely difficult. Much of the modern theory of dynamical systems is an tttempt to discover a residual set of "simple" dynamical systems - ones which can be classified and which will preserve their essential features when subjected to purturbations (see Smale [28], Nitecki [23], Peixoto [25], Newhouse [20], Palis and Melo [24]).

Bifurcation theory is not as well understood. In dim $m > 2$, much attention has been devoted to the appearance of strange attractors through bifurcation from a homoclinic orbit. For a hyperbolic fixed point of a diffeomorphism, a point $q \neq p$ is <u>homoclinic</u> to p if q is in the intersection of the stable and unstable manifolds of p. It is <u>transverse</u> <u>homoclinic</u> to p if the intersection is transversal. A special case of the results of Newhouse and Palis [21], [22] is concerned with one parameter families of diffeomorphisms $f(x,\lambda)$ which have the stable and unstable manifolds of p at $\lambda = 0$ tangent at q. Under certain conditions, they show there are families with the property that there is an infinite nonwandering set for each $\lambda < 0$ and the map is structurally stable for each $\lambda < 0$. Furthermore, there are infinitely many changes in the topological structure as $\lambda \to 0$. Newhouse [19] shows each of these systems also has infinitely many sinks.

The abstract theory can be realized for a planar differential equation with periodic coefficients

$$\ddot{y} + g(y) = -\lambda_1 \dot{y} + \lambda_2 f(t)$$

where $f(t+1) = f(t)$ and the equation for $\lambda_1 = \lambda_2 = 0$ has a homoclinic orbit. The appearance of homoclinic orbits for (λ_1, λ_2) in a neighborhood of zero has been discussed by several persons including Mel'nikov [17], Morozov [18], Holmes (see [11]), Chow, Hale, Mallet-Paret [7] and Hayashi and Ueda (see [33]). The most complete discussion is in [7] and relies only on classical concepts in differential equations. It is shown there are two curves in (λ_1, λ_2)-space which correspond to bifurcation to homoclinic points. In a neighborhood of any point on these curves, there are infinitely many subharmonic bifurcations, half being saddles and the others sinks. This example thus gives a concrete illustration of the abstract results in diffeomorphisms.

One also can observe this type of bifurcation in a more generic way by considering a planar equation

$$\dot{x} = X(x, \lambda) + \mu f(t, x)$$

where $f(t+1, x) = f(t, x)$ and the function $X(x, \lambda)$ at $\lambda = 0$ is a bifurcation point of degree one satisfying property (v) where a homoclinic orbit joins a saddle. For a generic class of f, one can adapt the methods in [7] to get the curve of bifurcation to homoclinic points. The theory of integral manifolds will imply the existence of a hyperbolic invariant torus $M_{\lambda,\mu}$ near the periodic orbit which bifurcated from the homoclinic orbit of $\dot{x} = X(x,\lambda)$. One can show there are subharmonic bifurcations on $M_{\lambda,\mu}$ as λ,μ vary.

As a final topic, we mention a recent result on the relationship between the dynamic behavior of a system and the bifurcation function obtained by the method of Liapunov-Schmidt. Consider the n-dimension equation

$$\dot{x} = Ax + f(t,x,\lambda) \tag{1}$$

where $f(t+1,x,\lambda) = f(t,x,\lambda)$, $f(t,0,0) = 0, \partial f(t,0,0)/\partial x = 0$, λ is a vector parameter, the matrix A has zero as a simple eigenvalue and the remaining ones with negative real parts. If we apply the method of Liapunov-Schmidt for the existence of 1-periodic solutions for (x,λ) near zero, we obtain a scalar bifurcation function $G(a,\lambda)$ depending on a scalar a and λ whose zeros correspond to 1-periodic solutions. Consider the scalar equation

$$\dot{a} = G(a,\lambda) \tag{2}$$

It is shown by deOliveira and Hale [9] that the stability properties of the equilibrium points of (2) determine the stability properties of the 1-periodic solutions of (1). The bifurcation function is relatively easy to approximate since it only requires equating Fourier series. It is certainly much easier than averaging techniques. More importantly, it permits a complete discussion of several parameter problems which averaging alone cannot do since it requires scaling of the parameters and an application of the implicit function theorem. The results apply directly to Hopf bifurcation since one can arrive at (1) by a change of variables with t representing an angle. Certain types of infinite dimensional evolutionary equations also can be discussed (see [9]).

The extent to which the above remarks are valid where A has a double eigenvalue zero certainly deserves extensive study. The problem is difficult because of the possibility of the appearance of invariant torii and homoclinic points. When one considers a three dimension autonomous equation near zero for which the linear part has a pair of purely imaginary eigenvalues and one zero eigenvalue, a change of variables leads to equation (1) in the plane with $A = 0$ and t an angle variable. Langford [13] has discussed some special bifurcations in this case.

REFERENCES

[1] A. A. ANDRONOV and F. A. LEONTOVICH, Sur la théorie de la variation de la structure qualitative de la division du plan en trajectoires. Dokl. Akad. Nauk 21(1938), pp. 427-430.

[2] A. A. ANDRONOV, F. A. LEONTOVICH, I. I. GORDON and A. G. MAIER, Theory of Bifurcations of Dynamical Systems on a Plane, Wiley, 1973.

[3] A. A. ANDRONOV and L. S. PONTRJAGIN, Grubye sistemy. Dokl. Akad. Nauk SSSR 14(1937), no.5.

[4] A. A. ANDRONOV, A. A. VITT, S. E. KHAIKIN, Theory of Oscillations, Pergamon 1966.

[5] J. CARR, Applications of Centre Manifold Theory, Lecture Notes, Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, June, 1979.

[6] N. CHAFEE, Generalized Hopf bifurcation and perturbation in a full neighborhood of a given vector field, Indiana Univ. Math. J. 27(1978).

[7] S-N. CHOW, J. K. HALE and J. MALLET-PARET, An example of bifurcation to homoclinic orbits, J. Diff. Eqn. Submitted.

[8] S-N. CHOW and R. WHITE, On the transition from supercritical to subcritical Hopf bifurcation.
To appear.

[9] J. C. deOLIVEIRA and J. K. HALE, Dynamic behavior from bifurcation equations. Tôhoku Math. J. To appear.

[10] D. FLOCKERZI, Bifurcation of Periodic Solutions from an Equilibrium Point. Dissertation, Wurzburg, 1979.

[11] P. HOLMES and J. E. MARSDEN, Qualitative techniques for bifurcation analysis of complex systems. Bifurcation Theory and Applications to Scientific Disciplines, N. Y. Acad. Sci. 1979.

[12] L. HOWARD and N. KOPPEL, Bifurcations and trajectories joining critical points. Adv. Math. 18(1976), pp. 306-358.

[13] W. F. LANGFORD, Periodic and steady state mode interactions lead to torii. SIAM J. Appl. Math. 37(1979), pp. 22-48.

[14] A. M. LIAPUNOV, Problème Général de la Stabilité du Mouvement, Princeton, 1949.

[15] A. M. LYAPUNOV, Sur les figures d'equilibre peu différentes des ellipsoides d'une masse liquide homogène donnée d'un mouvement de rotation. Zap. Akad. Nauk, St. Petersburg (1906).

[16] J. E. MARSDEN, Dynamical systems and invariant manifolds. Proc. New Approaches to Nonlinear Problems in Dynamics, Monterey, Calif. Dec. 9-14, 1979, SIAM Publications.

[17] V. K. MEL'NIKOV, On the stability of the center for time periodic solutions. Trans. Moscow Math. Soc. (Trudy) 12(1963), pp.3-56.

[18] A. D. MOROZOV, On the complete qualitative investigation of the equation of Duffing. Diff. Uravn. 12(1976), pp.241-255.

[19] S. NEWHOUSE, Diffeomorphisms with infinitely many sinks. Topology 12(1974), pp.9-18

[20] S. NEWHOUSE, Lectures on dynamical systems, C.I.M.E. Summer Session on Dynamical Systems, June 19-27, 1978.

[21] S. NEWHOUSE and J. PALIS, Bifurcations of Morse-Smale dynamical systems. Dynamical Systems, pp. 303-365, Academic Press, 1973.

[22] S. N"WHOUSE and J. PALIS, Cycles and bifurcation theory. Ast isque 31(1976), pp.43-141.

[23] Z. NITECKI, Differentiable Dynamics, M.I.T. Press, 1971.

[24] J. PALIS and W. MELO, Introducão aos Sistemas Dinamicos, IMPA, Rio de Janeiro, 1978.

[25] M. PEIXOTO, Dynamical Systems, Academic Press, 1973.

[26] H. POINCARÉ, Les Méthodes Norwelles de la Méchanique Céleste, Vol. 3, Gauthier-Villars, 1892.

[27] H. POINCARÉ, Sur l'équilibre d'une masse fluide animés d'un mouvement de rotation. Acta Math. 7(1885), pp.259-380.

[28] S. SMALE, Differentiable dynamical systems, Bull. Am. Math. Soc. 73(1967), pp.747-817.

[29] J. SOTOMAYOR, Generic one parameter families of vector fields on two dimensional manifolds. Pub. I.H.E.S. 43(1973).

[30] J. SOTOMAYOR, Structural stability and bifurcation theory, Dynamical Systems, pp. 549-560, Academic Press, 1973.

[31] J. SOTOMAYOR, Generic bifurcations and dynamical systems. Dynamical Systems, pp. 561-582, Academic Press, 1973.

[32] F. TAKENS, Forced oscillations and bifurcations. Communication 3 Inst. Rijksuniversiteit, Utrecht (1974).

[33] F. TAKENS, Singularities of vector fields. Publ. I.H.E.S. 43 (1974), pp. 47-100.

[34] Y. UEDA, Mapped forced oscillations by analog computer, Proc. New Approaches to Nonlinear Problems in Dynamics, Monterey, Calif. Dec. 9-14, 1979, SIAM Publications.